Implicit, Nonswitching, Vector-Oriented Algorithm for Steady Transonic Flow

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Introduction

IN many areas of aerodynamic technology, it is necessary to quickly compute a sequence of transonic flow solutions. For example, an adaptive-wall wind tunnel[1] requires solutions of the transonic small-disturbance equation for use in the control algorithm. Such on-line computational support would not be economically feasible if it were not for the low-cost vector array processors now available as peripherals for many minicomputers. In order to fully utilize the capability of the new hardware, algorithms must take advantage of their specialized architecture. The objective of this work is to develop an efficient algorithm for solving transonic flow problems governed by mixed partial differential equations on an array processor. The algorithm developed in this Note is designed with this application in mind; however, it should be equally applicable to large-scale, vector-oriented machines such as the CRAY 1 or CYBER-203/205 computers.

Murman and Cole[2] developed the first successive line over-relaxation (SLOR) algorithm for the solution of transonic flow problems. The difficulty of solving a mixed subsonic-supersonic flowfield was overcome by using a different stencil in each of the elliptic or hyperbolic regions. The small-disturbance potential equation used by Murman and Cole can be written in the following form:

$$(1-M^2)\phi_{xx}+\phi_{yy}=0 \tag{1}$$

where

$$M^2=M_\infty^2[1+(\gamma+1)\phi_x] \tag{2}$$

This nonlinear equation changes type when the coefficient of the ϕ_{xx} term changes sign. For the elliptic ($M<1$) part of the flow, the Murman-Cole algorithm uses central spatial differencing of ϕ_{xx}. For the hyperbolic ($M>1$) part of the flow, an upwind differencing of ϕ_{xx} is used.

Checking for subsonic or supersonic flow at each point by using the common FORTRAN "IF" branching statement does not allow vectorization of the algorithm; however, a transportable procedure for vectorizing the "IF" statement through the vectorizable "SIGN" function has been recently developed.[3] The FORTRAN statement

$$X=(A+B)+(A-B)*\mathrm{SIGN}(1,1-M^2)$$

is used where $X=2A$ if $1-M^2>0$ and $X=2B$ if $1-M^2<0$. This procedure requires more computational effort than the original Murman-Cole procedure, but allows vectorization.

A number of potentially more efficient iterative procedures have been proposed as alternatives to SLOR. Methods that have been used successfully to accelerate convergence for the small-disturbance equation include: 1) semidirect methods using fast Poisson solvers,[4] 2) approximate factorization schemes,[5] and 3) the multigrid approach.[6] However, all of these algorithms still use the type-dependent difference operators.

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The need for type-dependent differencing schemes in transonic flow calculations is a controversial issue. In recent papers, Holst and Ballhaus[7] and Steger and Caradonna[8] presented nontype-differencing (nonswitching) algorithms. Solving the unsteady, conservative, transonic full potential equation, Steger and Caradonna succeeded in getting a converged solution by computing the density with a second-order-accurate upwind differencing scheme over the entire flowfield. The ideas behind the present work are developed from the concepts described by Steger and Caradonna in the aforementioned paper.

Most of the SLOR and approximate factorization schemes in the two-dimensional case are designed to solve one line at a time, using some kind of tridiagonal solver, updating the potential, and proceeding to the next line. These types of algorithms are not easily vectorizable because of the order of dependency in the solving procedure. One of the few studies that dealt with the possibility of developing algorithms that take advantage of the vectorization ability of the new generation of vector-oriented computers in the solution of a system of tridiagonal matrices was described in a paper by South et al.[9] The algorithm developed in this Note uses an alternative SLOR method that enables the vectorization of the algorithm by simultaneously solving a system of pentadiagonal matrices.

Numerical Algorithm

Consider the problem of solving the simple steady, inviscid, two-dimensional, linear small-disturbance equation,

$$(1-M_\infty^2)\phi_{xx}+\phi_{yy}=0 \tag{3}$$

and let the difference operators ∇_x, Δ_x, ∇_y, and Δ_y be defined in the conventional way, for example, $\nabla_x\phi=(\phi_j-\phi_{j-1})/\Delta x$ and $\Delta_y\phi=(\phi_{k+1}-\phi_k)/\Delta y$.

It is well known that the difference schemes

$$(1-M_\infty^2)\nabla_x\Delta_x\phi+\nabla_y\Delta_y\phi=0 \tag{4}$$

and

$$(1-M_\infty^2)\nabla_x\nabla_x\phi+\nabla_y\Delta_y\phi=0 \tag{5}$$

are suitable for elliptic ($M_\infty<1$) and hyperbolic ($M_\infty>1$) problems, respectively. The differencing equation (4) is convergent for $M_\infty>1$ if $|\Delta x/[(1-M_\infty^2)\Delta y]|<1$, an impractical restriction for $M_\infty\to1$. Differencing equation (5) is divergent if $M_\infty<1$. To overcome these restrictions, Murman and Cole[2] introduced type-dependent difference operators in the transonic small-disturbance equation. In their approach, as described earlier, the streamwise spatial difference operators are switched from central to backward, depending on the local Mach number.

Various refinements to the Murman-Cole differencing have been introduced in the literature; however, usually overlooked is the fact that the model equations (4) and (5) can also be differenced, without any need to switch the difference operator, as follows (see Ref. 8):

$$\nabla_x\Delta_x\phi-M_\infty^2\nabla_x\nabla_x\phi+\nabla_y\Delta_y\phi=0 \tag{6}$$

By differencing Eqs. (4) and (5) in the alternative manner,

$$\nabla_x\Delta_x\phi-M_\infty^2\Delta_x\Delta_x\phi+\nabla_y\Delta_y\phi=0 \tag{7}$$

one gets an expansion "shock wave" near the leading edge, obviously a nonphysical solution.

The author was not able to prove convergence of Eq. (6) using the Von Neumann analysis. Nevertheless, it can be argued that by splitting the $(1-M_\infty^2)\phi_{xx}$ term of Eq. (3), as is done in Eq. (6), one eliminates the sharp distinction between the elliptic and hyperbolic character of Eq. (3). Equation (6)

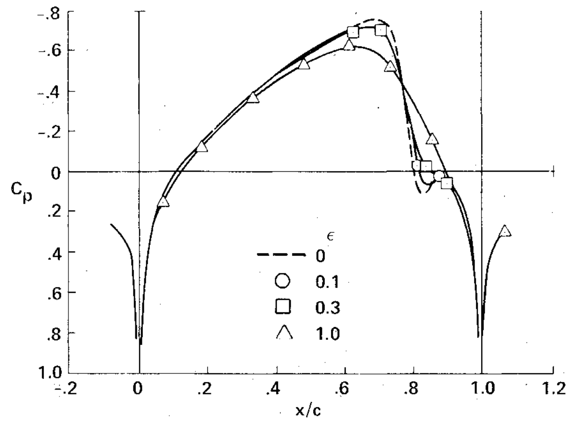

Fig. 1 Pressure distribution on a 10% biconvex airfoil at $M_\infty = 0.84$, computed by the small-disturbance equation.

can be split into two model equations,

$$\nabla_x \Delta_x \phi + \nabla_y \Delta_y \phi = 0 \, [\text{ignoring} - M_\infty^2 \nabla_x \nabla_x \phi] \tag{8}$$

$$-M_\infty^2 \nabla_x \nabla_x \phi + \nabla_y \Delta_y \phi = 0 \, [\text{ignoring} \, \nabla_x \Delta_x \phi] \tag{9}$$

Equations (8) and (9) are recognized as the Laplace and wave equations, respectively, which are stable for all Mach numbers. The sum of Eqs. (8) and (9) converges to give a physical solution, as demonstrated by Eq. (10).

In any event, using realistic boundary conditions, the algorithm given by Eq. (6) was evaluated through a large number of computational experiments and was found to be stable for a very fine grid (67 mesh points equally distributed on the chord).

The corresponding algorithm referred to here as MNDSD (mixed nonswitched differencing applied to the small-disturbance equation) that was derived to solve the small-disturbance equation [Eq. (1)] is given in Eq. (10) where the final approximation to the second derivative ϕ_{xx} is taken as a fraction ϵ of first-order and $(1-\epsilon)$ of second-order accuracy,

$$\nabla_x \Delta_x \phi - M^2 [\epsilon \bar{\phi}_{xx} + (1-\epsilon) \bar{\bar{\phi}}_{xx}] + \nabla_y \Delta_y \phi = 0 \tag{10}$$

where $\bar{\phi}_{xx} = \nabla_x \nabla_x \phi$ is a first-order backward difference approximation to the second derivative and $\bar{\bar{\phi}}_{xx} = \nabla_x \nabla_x (2 - E_x^{-1}) \phi$ is a second-order backward difference approximation to the second derivative, where $E_x^{\pm n} \phi_i = \phi_{i \pm n}$.

The ϕ_x term in the expression for M [Eq. (2)] was computed by central differences, but can be computed by backward or forward-backward differencing without influencing the solution. The scheme was found to be unstable for horizontal line relaxation, but stable for vertical line (flow direction). As was mentioned before, one complete iteration cycle consists of a sweep of the vertical odd lines solved simultaneously, followed by a sweep of all of the vertical even lines. In this alternative SLOR algorithm one has to solve a system of independent pentadiagonal matrices, a procedure that has the advantage of being vectorizable.

An analysis of the finite difference scheme for the $\bar{\phi}_{xx}$ and $\bar{\bar{\phi}}_{xx}$ shows that Eq. (1) represents the following partial differential equation:

$$\frac{\partial^2 \phi}{\partial x^2} - M^2 \frac{\partial^2 \phi}{\partial x^2} \frac{\partial^2 \phi}{\partial y^2} + M^2 \epsilon \Delta x \frac{\partial^3 \phi}{\partial x^3} + \text{HOT} = 0 \tag{11}$$

The ϕ_{xxx} term plays a special role in assuring convergence of the algorithm. It is usually referred to as an "artificial viscosity" that is added to the supersonic region to aid convergence. It will be shown that the algorithm converges for $\epsilon = 0$, which indicates that the ϕ_{xxx} term is not essential for convergence of the numerical scheme in the supersonic region. The fact that the algorithm converges for $\epsilon = 0$ means that the solution is second-order accurate throughout the flowfield.

A nonlifting airfoil problem is considered in order to demonstrate the utility of the algorithm. Tangency conditions are imposed on the (symmetrical) body surface with uniform flow at infinity. The surface boundary condition on $y = 0$ is

$$\phi_y = \frac{dy}{dx}\bigg|_{\text{airfoil}} ; \qquad \phi_y = 0 \bigg|_{\text{otherwise}} \tag{12}$$

where the surface is defined by $y = y(x)$. This boundary condition (BC) was implemented by updating the grid points at the level of the body ($k = 0$),

$$\phi_{j,0} = \phi_{j,1} - \text{BC} \,|_{\text{at} \; y=0} \tag{13}$$

where j is in the flow direction (x axis).

The computations were carried out on a coarse grid (33×21) and on two finer grids $(65 \times 21$ and $129 \times 21)$. Switching from one grid to another was done during the iteration process and was not based on the magnitude of the residual. In the finest grid, 67 points were equally distributed on the airfoil. The remaining streamwise points were spread unequally from 1.62 chords ahead to 2.04 chords aft of the airfoil. In the cross-flow direction, the mesh was stretched to an outer limit of 4.24 chords. This grid was used throughout the present work.

The first computational example is a biconvex, 10% thick airfoil at $M_\infty = 0.84$. Figure 1 shows the pressure distribution on the airfoil using different proportions of first- and second-order schemes for representing the streamwise second derivative term [Eq. (10)]. A pure ($\epsilon = 1$) first-order approximation for the backward ϕ_{xx} term smeared the shock beyond recognition. Using a greater fraction of the second-order approximation (ϵ decreasing) sharpens the shock. As mentioned before, the case for $\epsilon = 0$ is interesting because this corresponds to an example of a converged solution without "artificial viscosity" [see Eq. (11)]. It is also interesting to note that the postshock expansion is affected by the parameter ϵ.

The running time for a solution using 400 iterations in each of the three mesh levels was 24.4 CPU s on the CDC 7600 computer at NASA Ames Research Center and 3.7 CPU s on the CRAY 1 at Ames. The algorithm was used to solve the same airfoil at a higher Mach number ($M = 0.9$). Based on the same idea, a nonswitching algorithm was developed for a refined small-disturbance equation and a conservative nonswitching algorithm for the small-disturbance equation.

Conclusion

Implicit finite difference procedures were developed to solve the transonic small-disturbance equation in both conservative and nonconservative forms. A unified algorithm is introduced for solving the mixed transonic nonlinear flowfield equations. The alternative SLOR algorithm permits the vectorization of the pentadiagonal solver to take advantage of the special features of vector-oriented computers.

The method was tested on nonlifting airfoils and was found to be numerically stable, robust, and useful for rapid transonic flow computations.

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Applicability of the Independence Principle to Subsonic Turbulent Flow over a Swept Rearward-Facing Step

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Introduction

WITHIN a few years of each other, Prandtl,[1] Struminsky,[2], Jones,[3] and Sears[4] concluded that for yawed laminar incompressible flows the streamwise flow is independent of the spanwise flow. However, when the flow is turbulent, Ashkenas and Riddell[5] and Bradshaw[6] have reported that the "independence principle" (named by Jones) does not apply to yawed flat plates. Alternatively, on the basis of experiment, Young and Booth[7] and Altman and Hayter[8] have indicated that this principle is valid for the fully developed, turbulent boundary layer on yawed infinite cylinders. Theoretically, the Reynolds stress terms in the turbulent boundary-layer equations couple the spanwise and chordwise equations, precluding an independent solution.[9,10] However, for flow over finite geometries at small sweep angles (Λ), with corresponding small values of spatial spanwise velocity and derivatives, the "independence principle" may be applicable to many turbulent flows. As the sweep angle is increased, a sweep angle ("Λ_{crit}") is reached which defines the interval over which the "independence principle" is valid. The present results indicate the magnitude of Λ_{crit} for subsonic turbulent flow over a swept rearward-facing step.

Discussion

The present experiment was conducted in the NASA Langley subsonic low-turbulence open-loop wind tunnel. Splitter-plate models with step heights (h) of 0.32, 0.79, 1.27, and 2.38 cm and sweep angles of 0, 15, 30, 38, 45, and 60 deg were tested. A trip wire located 5.1 cm aft of the leading edge of the splitter plate insured the presence of a 2-cm thick turbulent boundary layer at the step. The Reynolds number (based on the distance at midspan between the leading edge and the step) varied from 7×10^5 to 2×10^6 ($11<V_\infty<32$ m/s).

Reattachment distance (R) data were obtained from oil flow patterns produced using the oil drop method. Visualization of the surface flow direction in the separated flow region downstream of a rearward-facing step is usually extremely difficult using the oil drop method because of the small magnitude of the surface shear stresses there. Black oil-based artist's paint thinned with linseed oil was applied to the surface downstream of the step in the form of droplets. Prior to this, a thin film of lightweight oil was spread over the downstream surface which had been painted white to provide a good contrast. The viscosity of the oil used to coat the surface and of the oil-paint mixture was varied until compatible values were identified through a trial-and-error process. If the viscosity of the dyed oil was too high, the droplets would not flow; if too low, the ground carbon particles would not remain in suspension. Similarly, if the

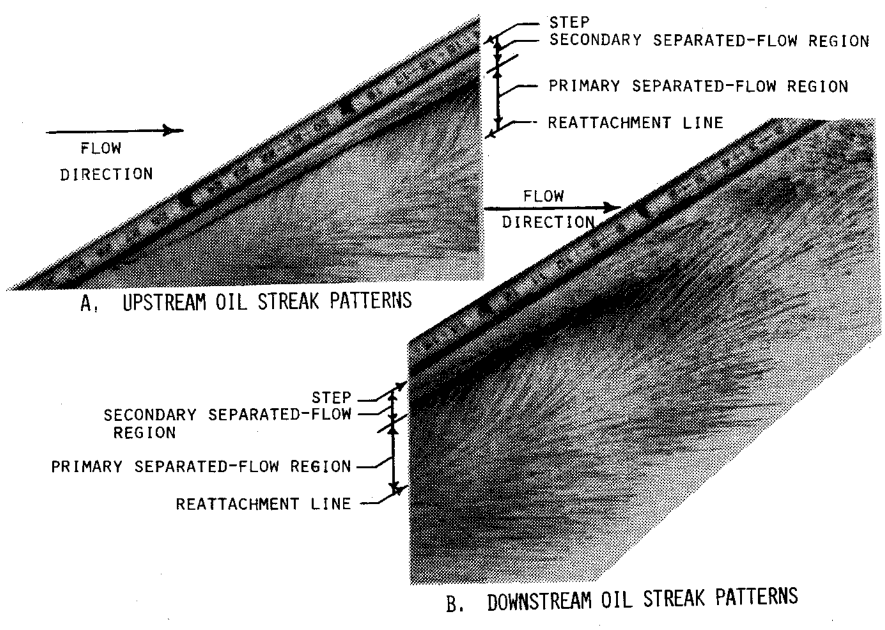

Fig. 1 Surface oil streak patterns ($\Lambda=60$ deg, $V_\infty=21$ m/s, and $h=2.38$ cm).

Received Aug. 2, 1982. This paper is declared a work of the U.S. Government and therefore is in the public domain.

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